to both formulae, but still leaves a decided difference in favor of the formula of Fullerton and Cattell (2.578 *vs.* 6.783).

(3) With practice the observed results approach the form of the Weber-Fechner Law and depart from the function of Fullerton and Cattell, so that the former hypothesis now becomes the more representative (25.527 *vs.* 20.180).[8]

(4) With the more limited range of intensities after practice the approximation to both hypotheses is greatly improved, as it was without practice; and the Weber-Fechner Law remains the more representative hypothesis, as it was after practice for the full range.

(5) If the equations of Table I are plotted together with the curves of the observed values, it will be found that the curve of actual values lies always between the two hypotheses; with practice, nearer the logarithmic curve; without practice, nearer the curve of the square root of the magnitude. Thus it appears that the formula of Fullerton and Cattell deviates from the logarithmic function of the Weber-Fechner Law in the proper direction, but deviates too far. Whether after great practice with Sanford's weights any correction of the Weber-Fechner formula in the direction of the equation of Fullerton and Cattell would be necessary, cannot be said.[9]

XXXVIII. An Example of the Fractionation of Data From the Method of Constant Stimuli for the Two-Point Limen

By L. B. Hoisington

We present herewith the results of a determination of the limen of dual impression upon the skin, by the method of constant stimuli,[1] in which the data have been fractionated and dealt with separately for every fraction. Our problem is analogous to that undertaken by Fernberger for lifted weights (*The Effects of Practice in Its Initial Stages in Lifted Weight Experiments and Its Bearing upon Anthropometric Measurements*).[2] We have, however, fractionated more completely than Fernberger, and have thus been able to extend our treatment to more general conclusions.[3] On the other hand, we have intended our case to be merely an illustration of the manner in which such data may be treated, for we used but two observers; whereas Fernberger had ten subjects.

The experimental work was performed by A. M. Palmer and P. R. Dickinson, both of whom accepted opportunities to enter the National Service at a time when the computation of their results was

[8]Titchener notes that the Weber-Fechner law is approached with practice; *op. cit.*, pt. ii, 83.

[9]We have noted that Plateau's law will approximate the observations, but that with so few determinations this approximation indicates little. Merkel's law (*cf.* Titchener, *op. cit.*, pt. ii, 69; Wundt, *op. cit.* I, 635 ff.), the straight line, is obviously inappropriate.

[1] E. B. Titchener, *Experimental Psychology*, II, 1905, i, 92-104; ii, 248-258; E. G. Boring, this Journal, 28, 1917, 280-293.

[2]S. W. Fernberger, this Journal, 27, 1916, 261-272.

[3] See Boring, this Journal, 27, 1916, 315-319; Fernberger, *Psychol. Bull.*, 14, 1917, 110-113; Boring, this Journal, 28, 1917, 280-293.

hardly more than begun. The writer is, therefore, indebted to these experimenters for data discussed in the present paper.

P and *D* worked together, the one acting as observer when the other experimented. They worked upon the forehead with the Griesbach aesthesiometer, using blunted conical rubber points applied at a pressure of 10g. The observer lay upon a couch; after every fifth series he rose and moved about in order that he should not become sleepy. He made his judgments before the points were removed from the skin. His ability to make the kind of judgment required was assured by preliminary practice-series. The experiment proper was concluded in 15 sessions; 10 or 15 series were taken at every session. A session always began with 5 'warming-up' series, which were discarded. The instructions were as follow: "After the signal 'ready,' you will be given a cutaneous stimulus on the forehead. You are to make an immediate judgment of 'one' or 'two'. The judgment 'two' is to stand for two discrete impressions; all other impressions are to be judged as 'one.' Make every judgment independently, without reference to any preceding impression. Keep as constant an attitude as possible;[4] if for any reason your attitude changes, so that you are in doubt about a particular impression, ask to have it repeated."

Five values of stimulus were used: 10, 11, 12, 13, and 14 mm. The total number of series was 200. These, for purposes of fractionation, were divided into groups of 100, of 50, and of 25. For every one of these fractions (15 in all[5]) we have computed the limen and the measure of precision (h), using Urban's tables[6], the probable error of the limen[7], and the sum of the squares of the deviations of the actual percentages from the theoretical.

1. *Practice.* Table I shows the values of the limen for all fractions, and Table II the corresponding values of h. There is (in the regular sense of the term) no practice-effect. The limens for both *O*'s, on the contrary, increase slightly from the first 100 to the second 100. If we examine the 50's and the 25's we see that this increase is not regular. There may, nevertheless, be some general tendency toward increase which is partially obscured by the great variability that occurs when the number of cases is few. If we plot the curves for these limens, by groups of 25 series, we note a general tendency toward

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[4] On the advantage of a constant attitude in psychophysical judgments, see S. S. George, this JOURNAL, 28, 1917, 1-37, esp. 33-36.

[5] *I. e.*, one group of 200 series, two 100's, four 50's, and eight 25's.

[6] In the case of the total group of 200 series there were six instances of half percentages. For these percentages we interpolated in Urban's tables for P and γP by fitting a parabola through three adjacent points. The method is more accurate than a straight-line interpolation. It makes no difference in the case of the half-percentages whether the interpolated point lies between the first two or the last two of the three points. On a similar case of parabolic interpolation see F. M. Urban, *The Application of Statistical Methods to the Problems of Psychophysics*, 1908, 124 f.

We checked the additions by the use of a checking table that gives the sums of the five values of Urban's tables for every value of x and p. This table is now in process of compilation by Dr. G. J. Rich; it is to be hoped that he may soon be able to publish it.

[7] For the manner of computation of the probable error of a limen, account being taken of the weighted number of cases, see Boring, this JOURNAL, 27, 317.

TABLE I

LIMENS (CM.)

Obs. *D*.

	Grouping by Series			
	By 25's	By 50's	By 100's	Total 200
1	12.119	12.348	12.214	12.348
2	12.551			
3	12.234	12.109		
4	11.981			
5	12.822	12.501	12.485	
6	12.229			
7	12.395	12.483		
8	12.548			

Obs. *P*.

	Grouping by Series			
	By 25's	By 50's	By 100's	Total 200
1	11.648	11.538	11.537	11.642
2	11.434			
3	11.452	11.532		
4	11.599			
5	11.658	11.578	11.667	
6	11.498			
7	11.710	11.783		
8	11.826			

TABLE II

MEASUREMENT OF PRECISION (h)

Obs. *D*.

	Grouping by Series			
	By 25's	By 50's	By 100's	Total 200
1	.4052	.3994	.4498	.4638
2	.3987			
3	.5740	.4986		
4	.4391			
5	.3565	.4151	.4828	
6	.4857			
7	.5153	.5696		
8	.6136			

Obs. *F*.

	Grouping by Series			
	By 25's	By 50's	By 100's	Total 200
1	.4128	.3837	.3970	.4540
2	.3719			
3	.3368	.4013		
4	.4765			
5	.3966	.4185	.3620	
6	.4424			
7	.2420	.3223		
8	.3834			

increase which appears by inspection to be almost as marked as in the true practice-curve of h for Fernberger's Subject I.[8] If, finally, we compute by the method of least squares the most probable straight line for our two curves, we find a general tendency throughout the span of 200 series for the limen to increase. The increase thus indicated is for D 0.292 mm. (2.4%), and for P 0.267 mm (2.3%).[9]

The practice-effect upon h is not the same for the two O's when all eight of the groups of 25 are considered. Plotting the curves by series, as above, we find for D the expected increase in the measure of precision to be 0.161 (35%), while the corresponding value for P is —0.059 (13%), a slight decrease. If, however, we plot the curve for the first six of the 25's there is, as in the case of D, a noticeable increase; the amount is 0.086 (19%). This discrepancy is accounted for by the fact that the last sessions were held after P, together with several others of his college fraternity, had enlisted in the army. It is probable that the constancy of his attitude was thereby greatly disturbed.

2. *Dependence of a Limen upon the Number of Observations on which It is Based.* Table III shows the probable errors of the limens. Since these values vary inversely with the square root of the weighted number of cases, we get a decrease of the average probable error as the number of series is increased from 25 to 200. An eight-fold increase in number of series (25 to 200) reduces the probable error of the limen, in the case of D, to one-third (.1104 to .0378) and, in the case of P, to less than one-third (.1294 to .0378).

The question how many series we must take in determining a limen depends in part upon what we mean to do with the limen when we get it. If we intend, as is usual, to compare it with another limen, then we must take enough series to reduce its probable error to a value which will give to the difference between it and the limen with which we compare it the probable correctness that we desire. We can not settle mathematically what probable correctness we shall regard as significant; we must decide the question in accordance with our scientific judgment in the particular case. Probably most psychologists would at present regard a probable correctness of 98% as very significant and one of 75% as not very significant.[10]

Suppose, for example, that we should wish to compare a limen taken from D with another limen which shows on the average the same degree of precision. If we found a difference of .484 mm. between the limens, then we can show by the proper computations that 25 series would be enough to establish this difference with a probable correctness of 98%. If the difference were only .165 mm., 200 series must needs have been taken to give a probable correctness of 98%. But 25 series would give for a difference of about .165 (exactly .156 mm.) a probable correctness of 75%. It appears, therefore, under our assumed condition, that 200 series would give a probable correctness of about 98% where 25 series would give a probable correctness of about 75%. It also appears that highly significant differences could be obtained from less than the usual 100 series if the difference were at all large.

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[8] Fernberger, *Psychol. Rev. Monog.*, No. 61, 1913, 42.

[9] For a general discussion of the effect of practice upon the limen for dual impression, see G. M. Whipple, *Manual of Mental and Physical Tests*, I, 1914, 254 f. and Bibliography; esp. L. Solomons, *Psychol. Rev.*, 4, 1897, 246-250; G. Tawney, *Philos. Stud.*, 13, 1897, 163-222.

[10] This JOURNAL, 28, 1917, 459.

TABLE III

PROBABLE ERRORS OF THE LIMEN

Obs. *D.*

<table>
<tr><th rowspan="2"></th><th colspan="4">Grouping by Series</th></tr>
<tr><th>By 25's</th><th>By 50's</th><th>By 100's</th><th>Total 200</th></tr>
<tr><td>1</td><td>.1187</td><td rowspan="2">.0862</td><td rowspan="4">.0539</td><td rowspan="8">.0378</td></tr>
<tr><td>2</td><td>.1262</td></tr>
<tr><td>3</td><td>.0932</td><td rowspan="2">.0719</td></tr>
<tr><td>4</td><td>.1116</td></tr>
<tr><td>5</td><td>.1352</td><td rowspan="2">.0830</td><td rowspan="4">.0532</td></tr>
<tr><td>6</td><td>.1036</td></tr>
<tr><td>7</td><td>.1001</td><td rowspan="2">.0658</td></tr>
<tr><td>8</td><td>.0947</td></tr>
<tr><td>Average</td><td>.1104</td><td>.0742</td><td>.0535</td><td>.0378</td></tr>
</table>

Obs. *P.*

<table>
<tr><th rowspan="2"></th><th colspan="4">Grouping by Series</th></tr>
<tr><th>By 25's</th><th>By 50's</th><th>By 100's</th><th>Total 200</th></tr>
<tr><td>1</td><td>.1168</td><td rowspan="2">.0882</td><td rowspan="4">.0604</td><td rowspan="8">.0369</td></tr>
<tr><td>2</td><td>.1283</td></tr>
<tr><td>3</td><td>.1395</td><td rowspan="2">.0848</td></tr>
<tr><td>4</td><td>.1057</td></tr>
<tr><td>5</td><td>.1220</td><td rowspan="2">.0826</td><td rowspan="4">.0709</td></tr>
<tr><td>6</td><td>.1123</td></tr>
<tr><td>7</td><td>.1869</td><td rowspan="2">.1006</td></tr>
<tr><td>8</td><td>.1244</td></tr>
<tr><td>Average</td><td>.1294</td><td>.0891</td><td>.0656</td><td>.0369</td></tr>
</table>

Suppose again, to make the illustration more concrete, that we consider the probable correctness of an individual difference between *D* and *P*. In Table IV we have computed[11] the probable correctness of

TABLE IV
PROBABILITY OF DIFFERENCE

	Probable Correctness of Difference					
	Average Difference		Maximal Difference		Minimal Difference	
Series	$\frac{D}{P.E._D}$	P. C.	$\frac{D}{P.E._D}$	P. C.	$\frac{D}{P.E._D}$	P. C.
200	13.55	100.00				
100	8.84	100 00	11.77	100.00	6.14	100.00
50	6.27	100.00	8.17	100.00	2.64	96.25
25	4.45	99.75	7.45	100.00	.93	73.36

the difference between the limens of *D* and *P* when the entire 200 series are taken, and when the minimal, maximal, and average differences are taken for the groups of 100 series, 50 series, and 25 series. We find that the probable correctness is 100.00% ("mathematical certainty") for the difference when 200 series are taken, and for any difference, even the minimal, between limens based on 100 series. If but 50 series are taken, we have 100.00% probable correctness for the difference between the averages of every set of four limens in the case of the minimal difference (between *D*'s 2nd 50, and *P*'s 4th 50), however, the probable correctness is 96.25%: a highly significant value, although not "mathematical certainty." Twenty-five series give 100.00% for the maximal differences, 99.86% for the averages, and only 73.36% for the minimal difference (between *D*'s 4th 25 and *P*'s 8th 25). If we regard 96% probable correctness as sufficiently significant (as most investigators would), we are justified in expecting that 50 series would be sufficient to establish the fact of an individual difference between *D* and *P*. Twenty-five series would usually give a difference of high probable correctness, yet not always, for the minimal difference is but 73%. The application to Fernberger's problem[12] is obvious; our individual difference is established by fifty series, not because practice-effect is considerably diminished, but because the number of observations attests the reliability of the indicated difference.

3. *The Approximation of the Actual Data to the Phi-Gamma Hypothesis.* Table V shows the sums of the squares of the deviations of the actual percentages from the theoretical percentages as determined under the Φ (γ)-hypothesis. The numbers, therefore, consti-

[11] The manner of computation of the values of Table IV as well as of the immediately preceding theoretical values is obvious from Boring, this JOURNAL, 27, 315-319.
[12] This JOURNAL, 27, 1916, 261 ff.

tute measures of the degree in which this hypothesis is realized in practice. The averages, for *D*, show that doubling the number of series tends approximately to halve the departure of the empirical data from the hypothesis. The same tendency is observable in the averages of *P*. The averages of the groups of 25, 50, and 100 series show a highly uniform ratio of decrease; here the averaging covers up the variations due to the unfavorable conditions already mentioned. It is clear, however, especially with *D*, that we approach the theoretical function as we increase the number of observations. This relationship suggests that the Φ (γ)-hypothesis is the correct hypothesis for these conditions, and that deviations from it are due in actual cases not to errors of theory but to errors of observation.[13]

The variability (M. V.) of these deviations also tends roughly to vary inversely with the number of the series. Two of the values for the groups of 25 are less than the value for the total group of 200 (*D*), although the average for the 25's is almost six times the value for the 200. For *P* the same relationship holds, although it is less extreme.

For *D* the Φ (γ)-hypothesis appears to be approached with practice. The departure of the second 100 from theory is only about one-eighth that of the first 100. Even when the errors of observation are increased by reduction of the number of the series to 50, the last 50 fit theory over five times more closely than the first 50. *P*'s results, however, show exactly the opposite relation.

TABLE V

SUMS OF THE SQUARES OF THE DEVIATIONS OF THE ACTUAL FROM THE THEORETICAL PERCENTAGES

Obs. *D*.

<table>
<tr><th rowspan="2"></th><th colspan="4">Grouping by Series</th></tr>
<tr><th>By 25's</th><th>By 50's</th><th>By 100's</th><th>Total 200</th></tr>
<tr><td>1</td><td>.012446</td><td rowspan="2">.025707</td><td rowspan="4">.007375</td><td rowspan="8">.002918</td></tr>
<tr><td>2</td><td>.070011</td></tr>
<tr><td>3</td><td>.001276</td><td rowspan="2">.002172</td></tr>
<tr><td>4</td><td>.004862</td></tr>
<tr><td>5</td><td>.020106</td><td rowspan="2">.006046</td><td rowspan="4">.000921</td></tr>
<tr><td>6</td><td>.002699</td></tr>
<tr><td>7</td><td>.015358</td><td rowspan="2">.000459</td></tr>
<tr><td>8</td><td>.009828</td></tr>
<tr><td>Average</td><td>.017073</td><td>.008596</td><td>.004148</td><td>.002918</td></tr>
<tr><td>M.V.</td><td>.013992</td><td>.008555</td><td>.003227</td><td></td></tr>
</table>

[13] Here we have a hint as to a method of separating errors of observation from errors of theory; *cf.* Urban, *Psychol. Rev.*, 17, 1910, 231 f.

Obs. *P*.

	Grouping by Series			
	By 25's	By 50's	By 100's	Total 200
1	.006310	.000199	.000537	.005683
2	.007635			
3	.024527	.003006		
4	.007560			
5	.025026	.019722	.005736	
6	.023411			
7	.030874	.005758		
8	.010099			
Average	.016934	.007173	.003142	.005683
M.V.	.009031	.006275	.002607	

With *D* practice is the more effective of these two factors. We approximate theory more closely in the second 100 than in the entire 200; practice more than makes up for the reduction of the number of series to one-half. The fourth 50 actually fit theory about seven times more closely than do the entire 200. If approximation to theory were, as it appears to be for *D*, a measure of the elimination of errors of observation and hence of the reliability of data, then we should do well to take many series, and to discard the early ones as merely furnishing practice.

Unlike *D*, however, *P* departs, as we have seen, farthest from the hypothesis in the last quarter of his work. This result may mean, on the face of it, that the hypothesis is not ideal for *P*; more probably however, certain disturbing factors increased in effectiveness during the course of the experiment.[14]

Finally, we may note that the absolute size of these measures of the deviation of practice from theory is comparable with the size of the measures upon which Urban assumes that the Φ (γ) is a better hypothesis for lifted weights than is the arctan-hypothesis.[15] One-third of our values (when divided by 3, the number of observations by which the limen is overdetermined) are less than the average of Urban's values (when divided by 5, the amount of overdetermination in Urban's case).[16]

[14] The situation which finally resulted in P's departure from the university for military service appears to have unfitted him for observation; see above, p. 592.

[15] *Op. cit.*, 258.

[16] *Cf.* the formulae in W. W. Johnson, *Theory of Errors and Least Squares*, 1915, 108; Urban, *Arch. f. d. ges. Psychol.*, 16, 1909, 226.